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Nonstandard Analysis in Classical Physics and Quantum Formal Scattering

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After a rigorous introduction to hyperreal numbers, we give in terms of non standard analysis, (1) a Lagrangian statement of classical physics, and (2) a statement of formal quantum scattering.

1. INTRODUCTION

In recent decades the notions of infinitesimal and infinite numbers have entered the dominion of pure mathematics, essentially due to the mathematician A. Robinson (see, for example, Robinson; 1974). This has given rise to a wide series of studies meant to clarify the historical relations between traditional analysis (now called "standard") and the approach now called "nonstandard analysis" (NSA). (See, e.g., the "Nonstandard Models" section of *Mathematical Reviews.*)

Not so numerous, but certainly more interesting for the physicist, is the research showing the undeniable formal and sometimes substantial advantages that the adoption of NSA offers to theoretical physics (see, e.g., Keleman, 1974; Helms and Loeb, 1979; Francis, 1981).

In the present paper we are interested primarily in the advantages of the nonstandard formulation rather than in its contents. We are persuaded in fact that the clearer synthesis that NSA permits provides many new starting points for theoretical physics.

This work is intended as the first step of a wider investigation: while here we test the general advantages of NSA on both classical and quantum grounds, further work is in progress to study Dirac's δ-function, and a third stage is intended in which we want to examine the same foundations of quantum mechanics using our new point of view.

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The paper, after a mathematical preface dealing with hyperreal numbers, is divided into two applications of NSA to physics: the first is a synthetic but complete reformulation of classical physics from the Lagrangian point of view; the second glances at a well-known method of standard scattering theory.

2. HYPERREAL NUMBERS

The aim of this section is to discuss a simple model of hyperreal numbers. We refer to Valenti (1982–1987) for further details.

2.1. Let E be a nonempty set, $P(E)$ the family of the parts of E, and m'' an external measure on $P(E)$.

With obvious and nonrestrictive assumptions on m'', the family $O(E)$ of the subsets with a zero measure has the following properties:

(a) $m''(0)=0$.
(b) $m''(E)\neq 0$.
(c) $\forall F\in P(E)$ and $\forall G\in P(E)$, if $m''(F)=0$ and $G\subset F\Rightarrow m''(G)=0$.
(d) $\forall F\in P(E)$ and $\forall G\in P(E)$, if $m''(F)=m''(G)=0\Rightarrow m''(F\cup G)=0$.

Generally, however, it is not true that

(e) $\forall F\in P(E)$ and $\forall G\in P(E)$, if $m''(F\cap G)=0\Rightarrow m''(F)\cdot m''(G)=0$.

If family $O(E)$ also has (e), it is called "a family of insignificant subsets."

2.2. Reference to an external measure helps give an intuitive justification for the introduced definition. It is clear, however, that we can consider propositions (a)–(e) in axiomatic form, postulating that some subset of E belongs to a generic family $O(E)$. This leads to considering a family $O(E)$ subject to the following requirements:

(a′) $\emptyset\in O(E)$.
(b′) $E\notin O(E)$.
(c′) $\forall F\subset E$, if $F\in O(E)$ and $G\subset F\Rightarrow G\in O(E)$.
(d′) $\forall F\subset E$ and $\forall G\subset E$, if $F\in O(E)$ and $G\in O(E)\Rightarrow F\cup G\in O(E)$.

Analogously, property (e) will be stated as follows:

(e′) $\forall F\subset E$ and $\forall G\subset E$, if $F\notin O(E)$ and $G\notin O(E)\Rightarrow F\cap G\notin O(E)$.

This is equivalent to

(e″) $\forall F\subset E$ and $\forall G\in E$, if $F\cap G\in O(E)\Rightarrow F\in O(E)$ or $G\in O(E)$.

Having disconnected our treatment from the concept of external measure, we consider the following:

Definition. Given a nonempty set E and a family $O(E)$ of parts of E, we will say that $O(E)$ is "of insignificant subsets" if it satisfies the following requirements:

(i) $\varnothing \in O(E)$.
(ii) $E \notin O(E)$.
(iii) $[F \in O(E) \wedge G \subset F] \Rightarrow G \in O(E)$.
(iv) $[F \in O(E) \wedge G \in O(E)] \Rightarrow F \cup G \in O(E)$.
(v) $F \cap G \in O(E) \Rightarrow [F \in O(E) \vee G \in O(E)]$.

In line with this definition we will call "significant" all the subsets of E that *do not* belong to $O(E)$.

Now, given a generic subset F of E and calling $E-F$ its complement with respect to E, we have the following:

Lemma. F belongs to $O(E)$ if and only if $E-F$ does not belong to $O(E)$, or

$$O(E)=\{F \in P(E): E-F \in P(E)-O(E)\}$$

Proof. Let $F \in O(E)$ and, *ab absurdo,* $E-F \in O(E)$; since $E=F \cup (E-F)$, property (iv) should imply $E \in O(E)$, which is contrary to (ii). On the other hand, let $E-F \in P(E)-O(E)$; since, because of (i), $(E-F) \cap F \in O(E)$, property (v) implies that at least one of $E-F$ and F belongs to $O(E)$, and this one can only be F.

This proves the lemma, which can now be expressed as follows: a subset of E is insignificant if and only if its complement is significant. ■

From the foregoing we have the following basic result:

Theorem. The family $S(E)=P(E)-O(E)$ of significant subsets satisfies the following properties:

(i) $\varnothing \notin S(E)$.
(ii) $E \in S(R)$.
(iii) $[F \in S(E) \wedge F \subset G] \Rightarrow G \in S(E)$.
(iv) $[F \in S(E) \wedge G \in S(E)] \Rightarrow F \cap G \in S(E)$.
(v) $F \cup G \in S(E) \Rightarrow [F \in S(E) \vee G \in S(E)]$.

We omit the proofs, based on the previous lemma.

Properties (I)-(V) characterize a very important mathematical structure: the "Ultrafilter."

2.3. To build our model, we now identify set E with the set N of natural numbers of which we consider first a generic finite subset G. The complement $N - G$ of this set will be called "cofinite." We denote by $F(N)$ the family of all the cofinite subsets of N.

It is easily verified that $F(N)$ satisfies requirements (I)-(IV), of previous paragraph, but not (V).

Nevertheless, it is possible to merge family $F(N)$ into another $S(N)$ (also constituted of subsets of N) that satisfies (I)-(V).

This interesting result ensures the existence within set N of (at least) one Ultrafilter containing the cofinite parts. This result (which we will not prove here) provides the key for a clear model of hyperreal numbers, as we will show next.

2.4. Let R be the real fileld, $R(N)$ the class of all the sequences of real numbers, and $S(N)$ a family of significant subsets of N containing the cofinite subsets. We give also the following:

Definition. Two sequences $\{a_n\}$ and $\{b_n\}$ of real numbers will be said to be "equivalent according to Robinson" (or "R-equivalent") if their elements coincide in correspondence with the naturals of a subset belonging to $S(N)$ (that is, with the indexes that constitute a significant set). It is easy to prove that this is an equivalence relation.

If we consider the set R^* of the classes in which $R(N)$ (which is the set of sequences of real numbers) is divided using $S(N)$, we can prove that: The set R^* of these equivalence classes, endowed with the natural sum and product operations, is a field.

Let us consider the following:

Definition. Let $\underset{\sim}{a}$ be an element of R^* and $\{a_n\}$ a sequence representative of $\underset{\sim}{a}$ such that $a_n > 0$ for every n of a significant set of indexes. We will then say that the element $\underset{\sim}{a}$ of field R^* is "positive" or "greater than $\underset{\sim}{0}$" (meaning by this the zero of R^*).

As this definition is independent of the choice of the representative $\{a_n\}$, it follows that: Set R^* can be endowed with a total order consistent with the field operations.

The ordered field R^* thus obtained is called the "field of hyperreal numbers."

2.5. Obviously R^* contains a subfield $R[\ ]$ isomorphic to the field R of real numbers. Its elements are called "real hyperreals" or, when there is no ambiguity, simply "reals."

We can also prove that in R^* there are:

1. Positive numbers smaller than every positive real
2. Negative numbers greater than every negative real
3. Positive numbers greater than every positive real

4. Negative numbers smaller than every negative real

Proof. We will prove only the first statement, that is, the one that ensures the existence of R^* of "infinitesimal positive numbers." Let $\underline{i}$ be the element of R^* represented, for example, by the sequence $\{i_n\} = 1, 1/2, 1/3, \ldots, 1/n, \ldots$, with n increasing in $N-\{0\}$. The indexes set $N-\{0\}$, being cofinite, is significant and therefore $\underline{i}$ is a positive hyperreal.

On the other hand, for any positive real x, we have $1/n < x$ for any n greater than a fixed $\bar{n}$ depending on x and thus every term of the $\{x - 1/n\}$ sequence is greater than 0 on a cofinite set of indexes, therefore significant.

Thus, hyperreal $\underline{x} - \underline{i}$, of which this sequence is representative, can only be positive. But x is arbitrary and therefore the statement is proved. ■

2.6. This legitimates the notions of "infinitesimal number," "finite number," and "infinite number." Moreover, for any finite hyperreal $\underline{a}'$, one and only one real hyperreal $\underline{a}$ infinitely close to it exists, such that difference $\underline{a}' - \underline{a}$ is infinitesimal. On this is based the following:

Definition. Let $\underline{a}'$ be a finite hyperreal number. We call the "standard part" of $\underline{a}'$, indicated by $\mathrm{st}(\underline{a}')$, the real hyperreal $\underline{a}$ infinitely close to $\underline{a}'$.

The difference $\underline{a}' - \mathrm{st}(\underline{a}')$ is called the "infinitesimal part" of $\underline{a}'$.

3. CLASSICAL PHYSICS

We will show how to get Lagrange equations using Hamilton's principle and the form that these equations taken in NSA. As is well known, the action of a classical system can be expressed as

$$I = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt \tag{1}$$

where q are generalized coordinates. The extremal value of I with respect to α, where

$$Q(t, \alpha) = q(t) + \alpha\chi(t) \tag{2}$$

yields

$$\left.\frac{\partial I}{\partial \alpha}\right|_{\alpha=0} = 0 \tag{3}$$

The NS form of the derivative is

$$\frac{\partial I}{\partial \alpha} = \int_{t_1}^{t_2} \left(\frac{\xi_Q}{\gamma_1} \frac{\gamma}{\delta} + \frac{\xi_{\dot{Q}}}{\eta_1} \frac{\eta}{\delta} \right) dt \tag{4}$$

where

$$\begin{aligned} \xi_Q &= L(Q+\gamma_1, \dot{Q}, t) - L(Q, \dot{Q}, t) \\ \xi_{\dot{Q}} &= L(Q, \dot{Q}+\eta_1, t) - L(Q, \dot{Q}, t) \\ \gamma &= Q(t, \alpha+\delta) - Q(t, \alpha) \\ \eta &= \dot{Q}(t, \alpha+\delta) - \dot{Q}(t, \alpha), \qquad \text{and} \qquad \gamma_1, \eta_1, \delta \in R^0 \end{aligned} \tag{5}$$

(R^0 is the set of infinitesimal numbers). From this and (2) one obtains

$$\frac{\gamma}{\delta} = \text{st}\left[\frac{Q(t, \alpha+\delta) - Q(t, \alpha)}{\delta}\right] = \chi(t) \tag{6}$$

and

$$\frac{\eta}{\delta} = \text{st}\left[\frac{\dot{Q}(t, \alpha+\delta) - \dot{Q}(t, \alpha)}{\delta}\right] = \dot{\chi}(t) \tag{7}$$

Using (6) and (7) in integral (4) yields

$$\frac{\partial I}{\partial \alpha} = \int_{t_1}^{t_2} \left[\frac{\xi_Q}{\gamma_1} - \left(\frac{\xi_{\dot{Q}}}{\eta_1}\right)\right] \chi \, dt$$

Moreover, (3) gives

$$\left.\frac{\partial I}{\partial \alpha}\right|_{\alpha=0} = \int_{t_1}^{t_2} \left[\frac{\xi_q}{\gamma_1} - \left(\frac{\dot{\xi}_{\dot{q}}}{\eta_1}\right)\right] \chi \, dt = 0 \tag{8}$$

which implies

$$\frac{\xi_q}{\gamma_1} - \left(\frac{\dot{\xi}_{\dot{q}}}{\eta_1}\right) = 0 \tag{9}$$

Thus, explicitly for n independent coordinates, the Lagrange equations take the following NS form:

$$\frac{\xi_{q_i}}{\gamma_i} - \left(\frac{\dot{\xi}_{\dot{q}_i}}{\eta_1}\right) = 0, \qquad i = 1, \ldots, n \tag{10}$$

where, for example,

$$\frac{\xi_{q_i}}{\gamma_1} = \text{st}\left[\frac{L(q_i+\gamma_1, q_{j\neq i}, \dot{q}_i, t) - L(q_i, \dot{q}_i, t)}{\gamma_1}\right]$$

and where the differential nature of the equations has been traded for a new NS algebraic nature.

In the same fashion, and using the Hamiltonian

$$H(q, p, t) = \sum_{i=1}^{N} \dot{q}_i p_i - L(q, \dot{q}, t) \tag{11}$$

where $p_i = \xi_{\dot{q}_i}/\eta_1$, we obtain Hamilton equations in the form

$$\dot{p}_i = -\eta_{q_i}/\gamma_1, \qquad \dot{q}_i = \eta_{p_i}/\beta_1, \qquad H'^{(t)} = L'^{(t)} \tag{12}$$

where

$$\eta_{q_i}/\gamma_1 = \mathrm{st}\left[\frac{H(q_i+\gamma_1, p_i, t) - H(q_i, p_i, t)}{\gamma_1}\right]$$

$$\eta_{p_i}/\beta_1 = \mathrm{st}\left[\frac{H(q_i, p_i+\beta_1, t) - H(q_i, p_i, t)}{\beta_1}\right]$$

$$H'^{(t)} = \mathrm{st}\left[\frac{H(t+\tau) - H(t)}{\tau}\right], \qquad \gamma_1, \beta_1, \tau \in R^0$$

As an example, consider a charged particle in an electromagnetic field. With obvious notation its Lagrangian is

$$L = \tfrac{1}{2}mv^2 - q\varphi(\mathbf{r}) + \frac{q}{c}\mathbf{v}\cdot\mathbf{A} \tag{13}$$

In NSA notation one immediately has

$$\frac{\boldsymbol{\xi}_{\mathbf{v}}}{\eta_1} = \mathrm{st}\left[\frac{L(\mathbf{r}, \mathbf{v}+\eta_1, t) - L(\mathbf{r}, \mathbf{v}, t)}{\eta_1}\right] = m\mathbf{v} + \frac{q}{c}\mathbf{A} \tag{14}$$

$$\frac{\boldsymbol{\xi}_{\mathbf{r}}}{\gamma_1} = -q\tilde{\boldsymbol{\nabla}}\varphi(\mathbf{r}) + \frac{q}{c}[(\mathbf{v}\cdot\tilde{\boldsymbol{\nabla}})\mathbf{A} + \mathbf{v}\wedge(\tilde{\boldsymbol{\nabla}}\wedge\mathbf{A})] \tag{15}$$

We remark that $\tilde{\boldsymbol{\nabla}}$ and $\tilde{\boldsymbol{\nabla}}\wedge$ operators in (15) are the NS extensions of grad and curl, and, as such, algebraic operators defined as

$$\tilde{\boldsymbol{\nabla}}\varphi(\mathbf{r}) = \mathrm{st}\left[\hat{\mathbf{i}}\frac{\Delta(\varphi_x)}{\gamma} + \hat{\mathbf{j}}\frac{\Delta(\varphi_y)}{\gamma} + \hat{\mathbf{k}}\frac{\Delta(\varphi_z)}{\gamma}\right]$$

$$\tilde{\boldsymbol{\nabla}}\wedge\mathbf{A}(\mathbf{r}) = \mathrm{st}\left[\hat{\mathbf{i}}\left(\frac{\Delta((A_z)_y)}{\gamma} - \frac{\Delta((A_y)_z)}{\gamma}\right) + \hat{\mathbf{j}}\left(\frac{\Delta((A_x)_z)}{\gamma} - \frac{\Delta((A_z)_x)}{\gamma}\right) + \hat{\mathbf{k}}\left(\frac{\Delta((A_y)_x)}{\gamma} - \frac{\Delta((A_x)_y)}{\gamma}\right)\right]$$

where, for example,

$$\Delta(\varphi_x) = \varphi(x+\gamma, y, z) - \varphi(x, y, z)$$

$$\Delta((A_z)_y) = A_z(x, y+\gamma, z) - A_z(x, y, z)$$

Finally, we get

$$\dot{\mathbf{A}} = (\mathbf{v} \cdot \tilde{\mathbf{\nabla}})\mathbf{A} + \mathbf{A}'^{(t)}$$

Using this last equality, substituting equations (14) and (15) in (10), and considering the canonical extensions to R^* of $\mathbf{E}$ and $\mathbf{B}$ as obtained from Maxwell equations yields the required Lorentz force:

$$\mathbf{F} = q(\mathbf{E} + 1/c \cdot \mathbf{v} \wedge \mathbf{B})$$

We emphasize that we have never used, implicitly or explicitly, the notion of limit, which is absolutely not essential in NSA.

4. QUANTUM PHYSICS

Here we discuss briefly a possible application of NSA to operators that in the standard approach are singular.

The resolvent operator

$$G(E) \equiv (E - H)^{-1} \tag{16}$$

where H is the hamiltonian of a physical system, is ubiquitous in quantum physics, and it plays an important role in such different contexts as relativistic quantum field theory, statistical physics, and the formal theory of scattering. Considered as a function of the c-number E, it displays singularities at the eigenvalues of H. This difficulty is normally circumvented by analytical continuation into the complex E-plane cut along the real axis.

The advantage presented by NSA with respect to the possibility of handling infinities as well as infinitesimals on the same footing as finite quantities permits a very natural way of outflanking the above difficulties.

Thus, assuming previous knowledge of the eigensolutions of

$$H_0|\phi_a\rangle = E_a^{(0)}|\phi_a\rangle \tag{17}$$

one is led to consider nonstandard, Hermitian operators H with NS hyperreal eigenvalues E, satisfying the Schrödinger equation

$$H|\psi_a\rangle = E_a|\psi_a\rangle, \qquad E_a = E_a^{(0)} + \zeta \quad \forall a, \qquad \text{with} \quad \zeta \in R^0 \tag{18}$$

Provided one can split the Hamiltonian as

$$H = H_0 + \lambda V, \qquad \lambda \in R^0, \qquad \lambda V = H_1 \tag{19}$$

it is easy to obtain

$$|\psi_a\rangle = \sum_n \lambda^n|n\rangle; \qquad \lambda^n|n\rangle = [G_0(E_a)H_1]^n|\phi_a\rangle \tag{20}$$

where

$$G_0(E_a) = (E_a - H_0)^{-1} \tag{21}$$

It is easy to show that by taking the standard part of (20) one obtains the limit for $\mathrm{Im}(E)\to 0$ of the solutions of the normal Lippmann-Schwinger equations.

It is interesting to remark that when the general procedure outlined above is specialized to the elementary case of potential scattering with

$$H_0=-\frac{\hbar^2}{2m}\tilde{\nabla}^2; \qquad H_1=V(\mathbf{r}); \qquad E=\frac{\hbar^2k^2}{2m}+\xi; \qquad \xi\in R^0 \tag{22}$$

where H_0 is a defined NS operator, a NSA version of the Green's function is obtained as

$$G(\mathbf{r},\mathbf{r}')=-\frac{1}{4\pi^2 i}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\int_{-\infty}^{+\infty}\frac{k'\exp(ik'|\mathbf{r}-\mathbf{r}'|)}{k^2-k'^2+\eta}\,dk' \tag{23}$$

where $\eta\in R^0$, which is formally very similar to the standard version, except for the fact that to all effects η in (23) is a NS real infinitesimal, which does not require any further mathematical step such as a limiting procedure. The integral (23) is now perfectly defined, but it is difficult to compute, so it is better to consider η as a complex infinitesimal.

It might be worth speculating about the possibility of developing an entirely algebraic NS integration procedure along appropriate paths in the R^* space, in analogy with the normal complex integration.

5. CONCLUSION

The above short mathematical summary and physical applications of NSA support the idea that the adoption of the latter is feasible in practice, and that it may in some cases lead to conceptual and formal, if not necessarily practical, simplifications. Further work along these lines is in progress.

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